

LIMIT

- If $a = \lim_{x \rightarrow 0} \frac{\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}}{x^4}$ and $b = \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$, then the value of $3a$ is :
 (1) 36 (2) 32
 (3) 25 (4) 30
- If $\lim_{x \rightarrow 0} \frac{3 + \alpha \sin x + \beta \cos x + \log_e(1-x)}{3 \tan^2 x} = \frac{1}{3}$, then $2\alpha - \beta$ is equal to :
 (1) 2 (2) 7
 (3) 5 (4) 1
- $\lim_{x \rightarrow 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{x^2}$
 (1) is equal to -1 (2) does not exist
 (3) is equal to 1 (4) is equal to 2
- Let $f: \mathbb{R} \rightarrow \mathbb{R} (0, \infty)$ be strictly increasing function such that $\lim_{x \rightarrow \infty} \frac{f(7x)}{f(x)} = 1$. Then, the value of $\lim_{x \rightarrow \infty} \left[\frac{f(5x)}{f(x)} - 1 \right]$ is equal to
 (1) 4 (2) 0
 (3) 7/5 (4) 1
- If $\lim_{x \rightarrow 0} \frac{ax^2 e^x - b \log_e(1+x) + cxe^{-x}}{x^2 \sin x} = 1$, then $16(a^2 + b^2 + c^2)$ is equal to _____.
- Let $\{x\}$ denote the fractional part of x and $f(x) = \frac{\cos^{-1}(1 - \{x^2\}) \sin^{-1}(1 - \{x\})}{\{x\} - \{x\}^3}$, $x \neq 0$. If L and R respectively denotes the left hand limit and the right hand limit of $f(x)$ at $x = 0$, then $\frac{32}{\pi} (L^2 + R^2)$ is equal to _____.
- Let $f(x) = \begin{cases} x-1, & x \text{ is even,} \\ 2x, & x \text{ is odd,} \end{cases} x \in \mathbb{N}$. If for some $a \in \mathbb{N}$, $f(f(f(a))) = 21$, then $\lim_{x \rightarrow a^-} \left\{ \frac{[x]^3}{a} - \left[\frac{x}{a} \right] \right\}$, where $[t]$ denotes the greatest integer less than or equal to t , is equal to :
 (1) 121 (2) 144
 (3) 169 (4) 225

- If $\alpha = \lim_{x \rightarrow 0^+} \left(\frac{e^{\sqrt{\tan x}} - e^{\sqrt{x}}}{\sqrt{\tan x} - \sqrt{x}} \right)$ and $\beta = \lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{2} \cot x}$ are the roots of the quadratic equation $ax^2 + bx - \sqrt{e} = 0$, then $12 \log_e(a+b)$ is equal to _____.
- $\lim_{x \rightarrow 0} \frac{e - (1+2x)^{\frac{1}{2x}}}{x}$ is equal to :
 (1) e (2) $\frac{2}{e}$
 (3) 0 (4) $e - e^2$
- If $\lim_{x \rightarrow 1} \frac{(5x+1)^{1/3} - (x+5)^{1/3}}{(2x+3)^{1/2} - (x+4)^{1/2}} = \frac{m\sqrt{5}}{n(2n)}$, where $\gcd(m, n) = 1$, then $8m + 12n$ is equal to _____
- If $1 + \frac{\sqrt{3} - \sqrt{2}}{2\sqrt{3}} + \frac{5 - 2\sqrt{6}}{18} + \frac{9\sqrt{3} - 11\sqrt{2}}{36\sqrt{3}} + \frac{49 - 20\sqrt{6}}{180} + \dots$ upto $\infty = 2 \left(\sqrt{\frac{b}{a}} + 1 \right) \log_e \left(\frac{a}{b} \right)$, where a and b are integers with $\gcd(a, b) = 1$, then $11a + 18b$ is equal to _____.
- Let $a > 0$ be a root of the equation $2^2x + x - 2 = 0$. If $\lim_{x \rightarrow \frac{1}{a}} \frac{16(1 - \cos(2 + x - 2x^2))}{(1 - ax^2)} = \alpha + \beta\sqrt{17}$, where $\alpha, \beta \in \mathbb{Z}$ then $\alpha + \beta$ is equal to _____.
- $\lim_{n \rightarrow \infty} \frac{(1^2 - 1)(n - 1) + (2^2 - 2)(n - 2) + \dots + ((n - 1)^2 - (n - 1)) \cdot 1}{(1^3 + 2^3 + \dots + n^3) - (1^2 + 2^2 + \dots + n^2)}$ is equal to :
 (1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) $\frac{3}{4}$ (4) $\frac{1}{2}$
- The value of $\lim_{x \rightarrow 0} 2 \left(\frac{1 - \cos x \sqrt{\cos 2x} \sqrt[3]{\cos 3x} \dots \sqrt[10]{\cos 10x}}{x^2} \right)$ is _____.

SOLUTIONS

1. Ans. (2)

Sol. $a = \lim_{x \rightarrow 0} \frac{\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}}{x^4}$
 $= \lim_{x \rightarrow 0} \frac{\sqrt{1+x^4} - 1}{x^4 (\sqrt{1+\sqrt{1+x^4}} + \sqrt{2})}$
 $= \lim_{x \rightarrow 0} \frac{x^4}{x^4 (\sqrt{1+\sqrt{1+x^4}} + \sqrt{2}) (\sqrt{1+x^4} + 1)}$

Applying limit $a = \frac{1}{4\sqrt{2}}$

$b = \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$
 $= \lim_{x \rightarrow 0} \frac{(1 - \cos^2 x)(\sqrt{2} + \sqrt{1 + \cos x})}{2 - (1 + \cos x)}$

$b = \lim_{x \rightarrow 0} (1 + \cos x)(\sqrt{2} + \sqrt{1 + \cos x})$

Applying limits $b = 2(\sqrt{2} + \sqrt{2}) = 4\sqrt{2}$

Now, $ab^3 = \frac{1}{4\sqrt{2}} \cdot (4\sqrt{2})^3 = 32$

2. Ans. (3)

Sol. $\lim_{x \rightarrow 0} \frac{3 + \alpha \sin x + \beta \cos x + \log_e(1-x)}{3 \tan^2 x} = \frac{1}{3}$
 $\Rightarrow \lim_{x \rightarrow 0} \frac{3 + \alpha \left[x - \frac{x^3}{3!} + \dots \right] + \beta \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right] + \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \right]}{3 \tan^2 x} = \frac{1}{3}$
 $\Rightarrow \lim_{x \rightarrow 0} \frac{(3 + \beta) + (\alpha - 1)x + \left[-\frac{1}{2} - \frac{\beta}{2} \right] x^2 + \dots}{3x^2} \cdot \frac{x^2}{\tan^2 x} = \frac{1}{3}$
 $\Rightarrow \beta + 3 = 0, \alpha - 1 = 0 \text{ and } \frac{-\frac{1}{2} - \frac{\beta}{2}}{3} = \frac{1}{3}$

$\Rightarrow \beta = -3, \alpha = 1$
 $\Rightarrow \alpha - \beta = 2 + 3 = 5$

3. Ans. (4)

Sol. $\lim_{x \rightarrow 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{x^2}$
 $\lim_{x \rightarrow 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{|\sin x|^2} \cdot \frac{\sin^2 x}{x^2}$
 Let $|\sin x| = t$
 $\lim_{t \rightarrow 0} \frac{e^{2t} - 2t - 1}{t^2} \cdot \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2}$
 $= \lim_{t \rightarrow 0} \frac{2e^{2t} - 2}{2t} \cdot 1 = 2 \cdot 1 = 2$

4. Ans. (2)

Sol. $f: \mathbb{R} \rightarrow (0, \infty)$

$\lim_{x \rightarrow \infty} \frac{f(7x)}{f(x)} = 1$

$\therefore f$ is increasing

$\therefore f(x) < f(5x) < f(7x)$

$\therefore \frac{f(x)}{f(x)} < \frac{f(5x)}{f(x)} < \frac{f(7x)}{f(x)}$

$1 < \lim_{x \rightarrow \infty} \frac{f(5x)}{f(x)} < 1$

$\therefore \left[\frac{f(5x)}{f(x)} - 1 \right]$

$\Rightarrow 1 - 1 = 0$

5. Ans. (81)

$ax^2 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) - b \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right)$
 $+ cx \left(1 - x + \frac{x^2}{x!} - \frac{x^3}{3!} + \dots \right)$

Sol. $\lim_{x \rightarrow 0} \frac{\dots}{x^3 \cdot \frac{\sin x}{x}}$

$= \lim_{x \rightarrow 0} \frac{(c-b)x + \left(\frac{b}{2} - c + a \right) x^2 + \left(a - \frac{b}{3} + \frac{c}{2} \right) x^3 + \dots}{x^3} = 1$

$c - b = 0, \frac{b}{2} - c + a = 0$

$a - \frac{b}{3} + \frac{c}{2} = 1 \quad a = \frac{3}{4} \quad b = c = \frac{3}{2}$

$a^2 + b^2 + c^2 = \frac{9}{16} + \frac{9}{4} + \frac{9}{4}$

$16(a^2 + b^2 + c^2) = 81$

6. Ans. (18)

Sol. Finding right hand limit

$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0+h)$

$= \lim_{h \rightarrow 0} f(h)$

$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-h^2) \sin^{-1}(1-h)}{h(1-h^2)}$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-h^2)}{h} \left(\frac{\sin^{-1} 1}{1} \right)$$

$$\text{Let } \cos^{-1}(1-h^2) = \theta \Rightarrow \cos \theta = 1-h^2$$

$$= \frac{\pi}{2} \lim_{\theta \rightarrow 0} \frac{\theta}{\sqrt{1-\cos \theta}}$$

$$= \frac{\pi}{2} \lim_{\theta \rightarrow 0} \frac{1}{\sqrt{\frac{1-\cos \theta}{\theta^2}}}$$

$$= \frac{\pi}{2} \frac{1}{\sqrt{1/2}}$$

$$R = \frac{\pi}{\sqrt{2}}$$

Now finding left hand limit

$$L = \lim_{x \rightarrow 0^-} f(x)$$

$$= \lim_{h \rightarrow 0} f(-h)$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-\{-h\}^2) \sin^{-1}(1-\{-h\})}{\{-h\} - \{-h\}^3}$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1-(-h+1)^2) \sin^{-1}(1-(-h+1))}{(-h+1) - (-h+1)^3}$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(-h^2+2h) \sin^{-1} h}{(1-h)(1-(1-h)^2)}$$

$$= \lim_{h \rightarrow 0} \left(\frac{\pi}{2} \right) \frac{\sin^{-1} h}{(1-(1-h)^2)}$$

$$= \frac{\pi}{2} \lim_{h \rightarrow 0} \left(\frac{\sin^{-1} h}{-h^2+2h} \right)$$

$$= \frac{\pi}{2} \lim_{h \rightarrow 0} \left(\frac{\sin^{-1} h}{h} \right) \left(\frac{1}{-h+2} \right)$$

$$L = \frac{\pi}{4}$$

$$\frac{32}{\pi^2} (L^2 + R^2) = \frac{32}{\pi^2} \left(\frac{\pi^2}{2} + \frac{\pi^2}{16} \right)$$

$$= 16 + 2$$

$$= 18$$

7. Ans. (2)

$$\text{Sol. } f(x) = \begin{cases} x-1; & x = \text{even} \\ 2x; & x = \text{odd} \end{cases}$$

$$f(f(f(a))) = 21$$

C-1: If a = even

$$f(a) = a-1 = \text{odd}$$

$$f(f(a)) = 2(a-1) = \text{even}$$

$$f(f(f(a))) = 2a-3 = 21 \Rightarrow a = 12$$

C-2: If a = odd

$$f(a) = 2a = \text{even}$$

$$f(f(a)) = 2a-1 = \text{odd}$$

$$f(f(f(a))) = 4a-2 = 21 \text{ (Not possible)}$$

Hence a = 12

Now

$$\lim_{x \rightarrow 12^-} \left(\left\lfloor \frac{|x|^3}{12} \right\rfloor - \left\lfloor \frac{x}{12} \right\rfloor \right)$$

$$= \lim_{x \rightarrow 12^-} \frac{|x|^3}{12} - \lim_{x \rightarrow 12^-} \left\lfloor \frac{x}{12} \right\rfloor$$

$$= 144 - 0 = 144$$

8. Ans. (6)

$$\text{Sol. } \alpha = \lim_{x \rightarrow 0^+} e^{\sqrt{x}} \left(\frac{e^{\sqrt{\tan x} - \sqrt{x}} - 1}{\sqrt{\tan x} - \sqrt{x}} \right)$$

$$= 1$$

$$\beta = \lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{2 \cot x}}$$

$$= e^{1/2}$$

$$x^2 - (1 + \sqrt{e}) + \sqrt{e} = 0$$

$$ax^2 + bx - \sqrt{e} = 0$$

On comparing

$$a = -1, b = \sqrt{e} + 1$$

$$12 n(a+b) = 12 \times \frac{1}{2} = 6$$

LIMIT

9. Ans. (1)

Sol.
$$\lim_{x \rightarrow 0} \frac{e - e^{\frac{1}{2x} \ln(1+2x)}}{x}$$

$$= \lim_{x \rightarrow 0} (-e) \left(\frac{e^{\frac{\ln(1+2x) - 1}{2x}} - 1}{x} \right)$$

$$= \lim_{x \rightarrow 0} (-e) \frac{\ln(1+2x) - 2x}{2x^2}$$

$$= (-e) \times (-1) \frac{4}{2 \cdot 2} = e$$

10. Ans. (100)

Sol.
$$\lim_{x \rightarrow 1} \frac{\frac{1}{3}(5x+1)^{-2/3} \cdot 5 - \frac{1}{3}(x+5)^{-2/3}}{\frac{1}{2}(2x+3)^{-1/2} \cdot 2 - \frac{1}{2}(x+4)^{-1/2}}$$

$$= \frac{8\sqrt{5}}{3 \cdot 6^{2/3}} \quad m=8, n=3$$

$$8m + 12n = 100$$

11. Ans. (76)

Sol.
$$S = 1 + \frac{x}{2\sqrt{3}} + \frac{x^2}{18} + \frac{x^3}{36\sqrt{3}} + \frac{x^4}{180} + \dots$$

Put $\frac{x}{\sqrt{3}} = t$, where $x = \sqrt{3} - \sqrt{2}$

$$S = 1 + \frac{t}{2} + \frac{t^2}{6} + \frac{t^3}{12} + \frac{t^4}{20} + \dots$$

$$S = 1 + t \left(1 - \frac{1}{2} \right) + t^2 \left(\frac{1}{2} - \frac{1}{3} \right) + t^3 \left(\frac{1}{3} - \frac{1}{4} \right) + t^4 \left(\frac{1}{4} - \frac{1}{5} \right) + \dots$$

$$S = \left(1 + t \left(\frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \dots \right) \right) - \left(\frac{t}{2} + \frac{t^2}{3} + \frac{t^3}{4} + \frac{t^4}{5} + \dots \right)$$

$$S = \left(t + \frac{t^2}{2} + \dots \right) - \frac{1}{t} \left(t + \frac{t^2}{2} + \frac{t^3}{3} + \dots \right) + 2$$

$$S = 2 + \left(1 - \frac{1}{t} \right) (-\log(1-t)) = \left(\frac{1}{t} - 1 \right) \log(1-t) + 2$$

$$S = 2 + \left(\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}} - 1 \right) \log \left(1 - \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}} \right)$$

$$S = 2 + \left(\frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}} \right) \log e^{\frac{\sqrt{2}}{\sqrt{3}}}$$

$$S = 2 + \frac{(\sqrt{6}+2)}{2} \log e^{\frac{2}{3}} = 2 + \left(\frac{\sqrt{3}}{\sqrt{2}} + 1 \right) \log e^{\frac{2}{3}}$$

$$a = 2, b = 3$$

$$11a + 18b = 11 \times 2 + 18 \times 3 = 76$$

12. Ans. (170)

Sol.
$$2x^2 + x - 2 = 0 \begin{matrix} \nearrow a \\ \searrow b \end{matrix}$$

$$2x^2 - x - 2 = 0 \begin{matrix} \nearrow a \\ \searrow \frac{1}{b} \end{matrix}$$

$$\lim_{x \rightarrow \frac{1}{a}} 16 \cdot \frac{\left(1 - \cos 2 \left(x - \frac{1}{a} \right) \left(x - \frac{1}{b} \right) \right)}{4 \left(x - \frac{1}{b} \right)^2} \cdot \frac{4 \left(x - \frac{1}{b} \right)^2}{a^2 \left(x - \frac{1}{a} \right)^2}$$

$$= 16 \cdot \frac{2}{a^2} \left(\frac{1}{a} - \frac{1}{b} \right)^2$$

$$= \frac{32}{a^2} \left(\frac{17}{4} \right) = \frac{17.8}{a^2} = \frac{17.8 \cdot 16}{(-1 + \sqrt{117})^2}$$

$$= \frac{136.16}{18.2\sqrt{7}} \cdot \frac{18 + 2\sqrt{7}}{18 + 2\sqrt{7}}$$

$$= \frac{136}{256} (18 + 2\sqrt{7}) \cdot 16$$

$$= 153 + 17\sqrt{7} = \alpha + \beta\sqrt{7}$$

$$\alpha + \beta = 153 + 17 = 170$$

13. Ans. (2)

Sol.
$$\lim_{n \rightarrow \infty} \frac{\sum_{r=1}^{n-1} (r^2 - r)(n-r)}{\sum_{r=1}^n r^3 - \sum_{r=1}^n r^2}$$

$$\lim_{n \rightarrow \infty} \frac{\sum_{r=1}^{n-1} (-r^3 + r^2(n+1) - nr)}{\left(\frac{n(n+1)}{2} \right)^2 - \frac{n(n+1)(2n+1)}{6}}$$

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{((n-1)n)}{2} \right)^2 + \frac{(n+1)(n-1)n(2n-1)}{6} - \frac{n^2(n-1)}{2}}{\frac{n(n+1)}{2} \left(\frac{n(n+1)}{2} - \frac{2n+1}{3} \right)}$$

$$\lim_{n \rightarrow \infty} \frac{\frac{n(n-1)}{2} \left(\frac{-n(n-1)}{2} + \frac{(n+1)(2n-1)}{3} - n \right)}{\frac{n(n+1)}{2} \frac{3n^2 + 3n - 4n - 2}{6}}$$

$$\lim_{n \rightarrow \infty} \frac{(n-1)(-3n^2 + 3n + 2(2n^2 + n - 1) - 6)}{(n+1)(3n^2 - n - 2)}$$

$$\lim_{n \rightarrow \infty} \frac{(n-1)(n^2 + 5n - 8)}{(n+1)(3n^2 - n - 2)} = \frac{1}{3}$$

14. Ans. (55)

Sol.

$$\lim_{x \rightarrow 0} 2 \left(\frac{1 - \left(1 - \frac{x^2}{2!}\right) \left(1 - \frac{4x^2}{2!}\right) \left(1 - \frac{9x^2}{2!}\right) \dots \left(1 - \frac{100x^2}{2!}\right)}{x^2} \right)$$

By expansion

$$\lim_{x \rightarrow 0} \frac{2 \left(1 - \left(1 - \frac{x^2}{2}\right) \left(1 - \frac{1 \cdot 4x^2}{2}\right) \left(1 - \frac{1 \cdot 9x^2}{3 \cdot 2}\right) \dots \left(1 - \frac{1 \cdot 100x^2}{2}\right) \right)}{x^2}$$

$$\lim_{x \rightarrow 0} 2 \left(\frac{1 - \left(1 - \frac{x^2}{2}\right) \left(1 - \frac{2x^2}{2}\right) \left(1 - \frac{3x^2}{2}\right) \dots \left(1 - \frac{10x^2}{2}\right)}{x^2} \right)$$

$$\lim_{x \rightarrow 0} \frac{2 \left(1 - 1 + x^2 \left(\frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \dots + \frac{10}{2} \right) \right)}{x^2}$$

$$2 \left(\frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \dots + \frac{10}{2} \right)$$

$$1 + 2 + \dots + 10 = \frac{10 \cdot 11}{2} = 55$$